Name: Solutions

Math 128A: Quiz 4

July 20, 2017

1. (6 points) Find the constants c_0, c_1, c_2 and x_1 such that the quadrature formula

$$\int_0^2 f(x)\ dx = c_0 f(0) + c_1 f(x_1) + c_2 f(2)$$

is exact for polynomials of as high a degree d as possible. Determine d as well.

Since there are four unknowns, we would expect the rule to be exact for at least $1, x, x^2$ and x^3. This yields

$$\int_0^2 1\ dx = 2 = c_0 + c_1 + c_2 \tag{1}$$

$$\int_0^2 x\ dx = 2 = c_1 x_1 + 2c_2 \tag{2}$$

$$\int_0^2 x^2\ dx = \frac{8}{3} = c_1 x_1^2 + 4c_2 \tag{3}$$

$$\int_0^2 x^3\ dx = 4 = c_1 x_1^3 + 8c_2 \tag{4}$$

Observe then that $(3) - 2 \times (2)$ and $(4) - 4 \times (2)$ give

$$c_1 x_1 (x_1 - 2) = -\frac{4}{3}, \quad c_1 x_1 (x_1^2 - 4) = -4. \tag{5}$$

Observe that $x_1 \neq 2$. Dividing the second equation by the first leads to

$$\frac{x_1^2 - 4}{x_1 - 2} = 3 \Rightarrow x_1 + 2 = 3 \Rightarrow \boxed{x_1 = 1}.$$

Plugging this in the first equation in (5) leads to $c_1(1)(1-2) = -4/3 \Rightarrow \boxed{c_1 = 4/3}$. Moreover, (2) yields $2 = \frac{4}{3} + 2c_2 \Rightarrow \boxed{c_2 = \frac{1}{3}}$. Finally, using these in (1) gives $2 = c_0 + \frac{4}{3} + \frac{1}{3} \Rightarrow \boxed{c_0 = \frac{1}{3}}$.

So far we have shown that the rule is exact for all polynomials of degree at most 3. For $f(x) = x^4$, however, we have

$$\int_0^2 x^4\ dx = \frac{32}{5} \neq \frac{20}{3} = \frac{4}{3}(1^4) + \frac{1}{3}(2^4)$$

so we conclude that $\boxed{d = 3}$.

2. (2 + 3 + 2 points) Consider $g(x) = \ln(x^2 + 1)$ on $[0, 3]$.

(a) Show that $|g^{(2)}(x)| \leq 2$ for all $x \in [0, 3]$.

We have $g'(x) = \frac{2x}{x^2+1}$ and

$$g^{(2)}(x) = \frac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \frac{2 - 2x^2}{(x^2+1)^2}.$$

We further have $g^{(3)}(x) = \frac{-4x(3-x^2)}{(x^2+1)^3}$ so the only stationary points of $g^{(2)}$ are at $x = 0, \pm\sqrt{3}$. Note that as $g^{(2)}(\sqrt{3}) = \frac{(2-2(3))}{(3+1)^2} = -\frac{1}{4}$, $g^{(2)}(0) = 2$ and $g^{(2)}(3) = \frac{2-2(9)}{(9+1)^2} = -\frac{16}{100}$, we can conclude that $|g^{(2)}(x)| \leq 2$ for all $x \in [0, 3]$.

(b) Recall the composite midpoint rule

$$\int_{x_{-1}}^{x_{n+1}} f(x)\, dx = 2h(f(x_0)+f(x_2)+f(x_4)+\ldots+f(x_{n-2})+f(x_n)) + \frac{h^2}{6}(b-a)f^{(2)}(c)$$

where $c \in (a, b)$. Use (a) to find the smallest number of sub-intervals required to estimate $\int_0^3 g(x)\, dx$ so that the error is at most 10^{-6}.

Note that the error term in absolute terms is $\frac{(3-0)^3}{6(n+2)^2}|g^{(2)}(c)|$ where $(n + 2)$ is the number of sub-intervals. We therefore require

$$\frac{(3^3)}{6(n+2)^2}(2) \leq 10^{-6} \Rightarrow (n+2)^2 \geq 3^2 \times 10^6 \Rightarrow n + 2 \geq 3 \times 10^3.$$

As 3000 is even, we conclude that the least number of sub-intervals is $\boxed{3000}$.

(c) Find the approximation to $\int_0^3 g(x)\, dx$ using the number of sub-intervals found in (b). You can leave your answer in the form of a sum but no unknown constants should appear in it.

Note that $n + 2 = 3000 \Rightarrow n = 2998$. Moreover, $h = \frac{(3-0)}{3000} = 10^{-3}$ and $x_i = x_{-1} + (i + 1)h = 10^{-3}(i + 1)$. The approximation to the integral then is

$$2h \sum_{i=0}^{n/2} g(x_{2i}) = \boxed{(2 \times 10^{-3}) \sum_{i=0}^{1499} \ln\left(1 + \frac{(2i+1)^2}{10^6}\right)}.$$

3. (2 + 3 + 2 points) Consider $f(x) = x^3 + x$ on $[0, 2]$ and let $I = \int_0^2 f(x)\, dx$.

(a) Apply the trapezoidal rule to approximate I using one and two sub-intervals; label these approximations S_1 and S_2 respectively.

We have

$$S_1 = \frac{(2-0)}{2}(f(0) + f(2)) = \boxed{10}$$

$$S_2 = \frac{(2-0)/2}{2}(f(0) + 2f(1) + f(2)) = \frac{1}{2}(0 + 2(2) + 10) = \boxed{7}.$$

(b) Recall that the composite trapezoidal rule provides an approximation of the form

$$I = N(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \dots. \tag{6}$$

Use your results from (a) and Richardson's extrapolation to find an $O(h^4)$ approximation to I.

Replacing h by $h/2$ in (6) gives

$$I = N(h/2) + K_1 \frac{h^2}{4} + K_2 \frac{h^4}{16} + \dots \tag{7}$$

$4 \times (7) - (6)$ then gives

$$3I = 4N(h/2) - N(h) - \frac{3K_2}{4} h^4 + \dots$$

so $N_2(h) = \frac{4}{3}N(h/2) - \frac{1}{3}N(h)$ is an $O(h^4)$ approximation to I.
From (a), we have $N(h) = S_1$ and $N(h/2) = S_2$ so the $O(h^4)$ approximation to I is $\boxed{\frac{1}{3}(4S_2 - S_1) = 6}$.

(c) Calculate I exactly and compare it to your answer from (b). Explain your result.

Note that $I = \int_0^2 x^3 + x\, dx = [x^4/4 + x^2/2]_0^2 \Rightarrow \boxed{I = 6}$.
This is identical to the "approximation" in (b). Recall that the error terms for composite integration rules are of the form $Kh^m f^{(m)}(c)$. As our result in (b) has an error that is $O(h^4)$, it should be exact if $f^{(4)} \equiv 0$ which is indeed the case for $f(x) = x^3 + x$.